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Some remarks on extraresolvable spaces

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Abstract

We give an example of a countable extraresolvable space that is not strongly extraresolvable. We also prove that $\mathbb{Q} \times \omega_1$ is strongly extraresolvable, and if X is strongly extraresolvable and $\mathrm{nwd}(X) = \Delta(X) = \omega$, then $X \times \omega$ is strongly extraresolvable (ω_1 and ω are equipped with the discrete topology). © 2002 Elsevier Science B.V. All rights reserved.

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0. Introduction

Throughout this paper, our spaces will be Tychonoff and crowded. The Greek letters α and κ denote infinite cardinal numbers. For an ordinal θ, θ will also stand for de discrete space of underlying set θ. The Stone–Čech compactification $\beta(\omega)$ of the natural numbers equipped with the discrete topology will be identified with the set of all ultrafilters on ω and its remainder $\omega^* = \beta(\omega) - \omega$ will be identified with the set of all free ultrafilters on ω.

Following Hewitt [12], we say that a space is *resolvable* if it contains two disjoint dense subsets. Metric spaces and locally compact spaces are examples of resolvable spaces [12], [5, Theorem 3.7]. It is not difficult to see that a space X cannot have more than $\Delta(X)$-many pairwise disjoint dense subsets. In 1964, Ceder [4] studied the class of spaces (called κ*-resolvable*) that contain k many pairwise disjoint dense subsets, for a cardinal $\kappa \geqslant 2$. A space X that is $\Delta(X)$-resolvable is called *maximally resolvable*. El'kin [10] proved that if $\pi w(X) \leqslant \Delta(X)$, then X is maximally resolvable (for more examples of maximally resolvable spaces see [5] and [16]). A space X is said to be *extraresolvable*

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if X contains a family $\mathcal{D}$ of dense subsets such that $|\mathcal{D}| > \Delta(X)$ and the intersection of any two distinct elements of $\mathcal{D}$ is nowhere dense, where $\Delta(X) = \min\{|U|\colon U$ is a nonempty open subset of $X\}$. We say that a space X is *strongly extraresolvable* if there is a family $\mathcal{D}$ of dense subsets of X such that $|\mathcal{D}| > \Delta(X)$ and $|D \cap E| < \mathrm{nwd}(X)$ whenever D and E are distinct elements of $\mathcal{D}$, where $\mathrm{nwd}(X) = \min\{|A|\colon A \subseteq X$, $\mathrm{int}_X(\mathrm{cl}_X A) \neq \emptyset\}$. The extraresolvability was introduced by Malykhin [15] and it is a generalization of ω-resolvability (see [1,2,6,7,11]). Every countable Frechét–Urysohn space is extraresolvable [11]. In particular, the rational numbers $\mathbb{Q}$ are extraresolvable. But the real line $\mathbb{R}$, is not $\mathfrak{c}$-resolvable and is not extraresolvable (for details see [11]). Later Comfort and García-Ferreira [6,7] introduced the notion of strongly extraresolvability which is a strengthening of extraresolvability. It is shown in [7] that if X is extraresolvable, then $X \times \alpha$ is always extraresolvable, for every infinite cardinal α, and if $X \times \alpha$ is strongly extraresolvable, then X is strongly extraresolvable and $\alpha \leqslant \Delta(X)^+$. Thus, $\mathbb{Q} \times \omega_2$ is extraresolvable and is not strongly extraresolvable. It is also proved in [7, Theorem 4.1(d')] that if X is strongly extraresolvable, then $X \times \alpha$ is strongly extraresolvable for every $\alpha < \mathrm{nwd}(X)$. Hence, we may ask: if $X \times \alpha$ is strongly extraresolvable must we have that $\alpha < \mathrm{nwd}(X)$. It turns out that the answer is no: In [7], the Comfort and García-Ferreira proved that $C_p([0, \beth_{\omega_1})) \times \beth_{\omega_1}$ is strongly extraresolvable, where $[0, \beth_{\omega_1})$ is equipped with the order topology, and $\Delta(C_p([0, \beth_{\omega_1}))) = \mathrm{nwd}(C_p([0, \beth_{\omega_1}))) = \beth_{\omega_1}$, where $\beth_0 = \omega$, $\beth_{\theta+1} = 2^{\beth_\theta}$ for all ordinal θ, and $\beth_{\omega_1} = \sum_{\theta<\omega_1} \beth_\theta$. In the present paper, we prove that $\mathbb{Q} \times \omega_1$ is strongly extraresolvable (notice that $\mathrm{nwd}(\mathbb{Q}) = \Delta(\mathbb{Q}) = \omega$). Thus, $\Delta(X)^+$ is the best upper bound of all cardinal numbers α for which $X \times \alpha$ is strongly extraresolvable. In the first section, we give an example of an extraresolvable countable space which is not strongly extraresolvable (this answers affirmatively Question (a) from [7]).

1. A countable extraresolvable space which is not strongly extraresolvable

Let $\mathbf{Seq} = \bigcup_{n<\omega} \omega^n$ be the set of all finite sequences of natural numbers. If $s \in \mathbf{Seq}$, then $l(s)$ will denote the length of s, and if $n < \omega$, then $s^\frown n$ will denote the finite sequence $s \cup \{(\mathrm{dom}(s), n)\}$. For every $s \in \mathbf{Seq}$, we chose a $q_s \in \omega^*$ and put $\mathcal{A} = \{q_s\colon s \in \mathbf{Seq}\}$. Then, we define a topology $\tau_\mathcal{A}$ on $\mathbf{Seq}$ by putting: $V \in \tau_\mathcal{A}$ iff for every $s \in V$, $\{n < \omega\colon s^\frown n \in V\} \in q_s$. It is known that $\mathbf{Seq}(\mathcal{A}) = (\mathbf{Seq}, \tau_\mathcal{A})$ is a Tychonoff, extremally disconnected, zero-dimensional, crowded space (see [9]). In what follows, $\mathcal{A}$ will always denote a family $\{q_s\colon s \in \mathbf{Seq}\}$ of elements of ω^*.

We shall use the notation from [14,9]. For $n < \omega$, we let $L_n = \{s \in \mathbf{Seq}\colon l(s) = n\}$, and $T_n = \bigcup_{m \leqslant n} L_m$ which is a closed nowhere dense subset of $\mathbf{Seq}(\mathcal{A})$. The cone of $s \in \mathbf{Seq}$ is the set $C(s) = \{t \in \mathbf{Seq}\colon s \subseteq t\}$, and if $f \in \prod_{t \in C(s)-\{s\}} q_t$, then $C(s, f) = \bigcup_{k<\omega} C_k$ is the cone over s with respect to the function f, where $C_0 = \{s\}$ and $C_{k+1} = \{t^\frown n\colon n \in f(t),\ t \in C_k\}$ for each $1 \leqslant k < \omega$. The set $\{C(s, f)\colon s \in \mathbf{Seq}$ and $f \in \prod_{t \in C(s)-\{s\}} q_t\}$ is a base for the topology of $\mathbf{Seq}(\mathcal{A})$.

Theorem 1.1. $\mathbf{Seq}(\mathcal{A})$ *is extraresolvable.*

Proof. Let $\Sigma \subseteq [\omega]^\omega$ be an infinite maximal almost disjoint family (i.e., $|A \cap B| < \omega$ whenever $A, B \in \Sigma$ and $A \neq B$). It is known that $|\Sigma| \geqslant \omega_1$ (see [8]). For each $A \in \Sigma$, let $D_A = \{s \in \mathbf{Seq}\colon\ l(s) \in A\}$. It is not hard to see that D_A is a dense subset of $\mathbf{Seq}(\mathcal{A})$ for every $A \in \Sigma$. If $A \neq B \in \Sigma$, then $D_A \cap D_B \subseteq T_m$, where $m = \max\{A \cap B\}$. Thus, $D_A \cap D_B$ is nowhere dense in $\mathbf{Seq}(\mathcal{A})$ whenever $A, B \in \Sigma$. Therefore, $\mathbf{Seq}(\mathcal{A})$ is extraresolvable. □

The next lemma gives a useful property of the dense subsets of $\mathbf{Seq}(\mathcal{A})$.

Lemma 1.2. *If D is dense in $\mathbf{Seq}(\mathcal{A})$, then $\forall s \in \mathbf{Seq}\, \exists t \in \mathbf{Seq}$ $(s \subseteq t \wedge \{n < \omega\colon\ t^\frown n \in D\}$ $\in q_t$.*

Proof. Suppose that there is $s \in \mathbf{Seq}$ such that for every $t \in \mathbf{Seq}$ with $s \subseteq t$, $\{n < \omega\colon\ t^\frown n \notin D\} \in q_t$. Define $f \in \prod_{t \in C(s)-\{s\}} q_t$ by $f(t) = \{n < \omega\colon\ t^\frown n \notin D\}$ for each $t \in C(s)$. It is then clear that $D \cap (C(s, f) - \{s\}) = \emptyset$, which is a contradiction. □

It is not hard to see that Lemma 1.2 does not characterize the dense subsets of $\mathbf{Seq}(\mathcal{A})$.

Lemma 1.3. $\mathbf{Seq}(\mathcal{A})$ *is not strongly extraresolvable.*

Proof. Suppose that $\{D_\nu\colon\ \nu < \omega_1\}$ witnesses the strong extraresolvability of $\mathbf{Seq}(\mathcal{A})$. In particular we have that, $|D_\nu \cap D_\mu| < \omega$ for every $\nu < \mu < \omega_1$. For each $\nu < \omega_1$, we let $S_\nu = \{t \in \mathbf{Seq}\colon\ \{n < \omega\colon\ t^\frown n \in D_\nu\} \in q_t\}$, which is nonempty by Lemma 1.2. As $\mathbf{Seq}$ is countable, there must be $\nu < \mu < \omega_1$ such that $S_\nu \cap S_\mu \neq \emptyset$. But, if $t \in S_\nu \cap S_\mu$, then $\{n < \omega\colon\ t^\frown n \in D_\nu \cap D_\mu\} \in q_t$ and so $D_\nu \cap D_\mu$ is an infinite, which is impossible since $D_\nu \cap D_\mu$ is finite. This shows that $\mathbf{Seq}(\mathcal{A})$ cannot be strongly extraresolvable. □

If q_s is just a free filter on ω for all $s \in \mathbf{Seq}$, then $\mathbf{Seq}(\mathcal{A})$ is a crowded zero-dimensional Tychonoff space. As a particular case, if $f_s = \{A \subseteq \omega\colon\ |\omega - A| < \omega\}$ is the Fréchet filter on ω, for every $s \in \mathbf{Seq}$, then $\mathbf{Seq}(\{f_s\colon\ s \in \mathbf{Seq}\})$ is homeomorphic to the space S_ω from [3] and, by Corollary 2.9 of [11], S_ω is strongly extraresolvable; hence, $\mathbf{Seq}(\{f_s\colon\ s \in \mathbf{Seq}\})$ is strongly extraresolvable. This shows that the hypothesis that q_s is a free ultrafilter is essential in our example.

2. Spaces $X \times \alpha$

To prove that $\mathbb{Q} \times \omega_1$ is strongly extraresolvable we need the following two lemmas.

Lemma 2.1. *Let $\pi w(X) = \omega$. If $\{D_n\colon\ n < \omega\}$ is a countable family of dense subsets of X such that $D_m \cap D_n$ is nowhere dense for each $m < n < \omega$, then for every $N < \omega$ there is a dense subset D of X such that $D \cap D_m$ is finite for every $m < \omega$ and $D \cap D_m = \emptyset$ for every $m < N$.*

Proof. Let $\{B_m\colon m<\omega\}$ be a countable π-base for X. It is evident that $D_n-(\bigcup_{j<n}D_j)$ is dense in X for every $n<\omega$. So, for each $m<\omega$, pick a point $d_m\in B_m\cap(D_{N+m}-(\bigcup_{j<N+m}D_j))$. Define $D=\{d_m\colon m<\omega\}$. Then D is a dense subset of X, $|D\cap D_n|<\omega$ for every $n<\omega$ and $D\cap D_n=\emptyset$ for every $n<N$. □

Theorem 2.2. *If $\pi w(X)=\omega$ and $\Delta(X)=\omega$, then $X\times\omega_1$ is strongly extraresolvable.*

Proof. As $\Delta(X)=\omega$, $\mathrm{nwd}(X)=\omega$. By Theorem 2.3 of [7], X is strongly extraresolvable, and, by the result of El'kin [10] mentioned above, there is a countable family $\mathcal{D}$ of pairwise disjoint dense subsets of X. Now, for each $\omega\leqslant\nu<\omega_1$ we fix a bijection $\sigma_\nu:\nu\to\mathcal{D}$ and we let $\{D^n_m\colon m,n<\omega\}$ be a pairwise disjoint, faithfully indexed, family of dense subsets of X. Now assume that for every $\nu,\mu<\theta<\omega_1$, a dense subset D^ν_μ of X has been defined so that

(1) $\{D^\nu_\mu\colon \nu\leqslant\mu\}$ is pairwise disjoint for every $\mu<\theta$;
(2) If $\lambda<\theta$, then $|\{\mu\leqslant\lambda\colon D^\nu_\mu\cap D^\lambda_\mu\neq\emptyset\}|<\omega$ for every $\nu<\lambda$;
(3) $|D^\nu_\mu\cap D^{\nu'}_\mu|<\omega$ for every $\mu,\nu,\nu'<\theta$ with $\nu\neq\nu'$.

Put $D^\nu_\theta=\sigma_{\theta+1}(\nu)$ for every $\nu\leqslant\theta$. Let us consider the space $X\times\theta$ and let $E_\nu=\bigcup_{\mu<\theta}(D^\nu_\mu\times\{\mu\})$ for $\nu<\theta$. Then $\{E_\nu\colon \nu<\theta\}$ is a countable family of dense subsets of $X\times\theta$ such that $E_\mu\cap E_\nu$ is finite for each $\mu<\nu<\theta$. As $\pi w(X\times\theta)=\omega$, by Lemma 1.1, there is a dense subset D of $X\times\theta$ such that $D\cap E_\nu$ is finite for every $\nu<\theta$. Then $D=\bigcup_{\mu<\theta}(D^\theta_\mu\times\{\mu\})$, where D^θ_μ is a dense subset of X. It is not hard to see that D^θ_μ satisfies conditions 1, 2 and 3, for every $\mu\leqslant\theta$. Finally, let $D_\nu=\bigcup_{\mu<\omega_1}(D^\nu_\mu\times\{\mu\})$ for each $\nu<\omega_1$. So, $\{D_\nu\colon \nu<\omega_1\}$ is a family of dense subsets of $X\times\omega_1$ such that $D_\mu\cap D_\nu$ is finite provided that $\mu<\nu<\omega_1$. □

Corollary 2.3. *$\mathbb{Q}\times\omega_1$ is strongly extraresolvable.*

The following result gives a necessary condition for strong extraresolvability of $X\times\alpha$.

Theorem 2.4. *If $X\times\alpha$ is strongly extraresolvable and $\mathrm{nwd}(X)<\alpha\leqslant\Delta(X)^+$, then X is κ-resolvable for every cardinal $\kappa<\alpha$. In particular, if $X\times\Delta(X)^+$ is strongly extraresolvable, then X is maximally resolvable.*

Proof. We remark that $\mathrm{nwd}(X)=\mathrm{nwd}(X\times\alpha)\leqslant\Delta(X)=\Delta(X\times\alpha)$. Let α be a cardinal number satisfying $\mathrm{nwd}(X)<\alpha\leqslant\Delta(X)^+$, and let $\mathcal{E}$ be a family of dense subsets of $X\times\alpha$ witnessing the strong extraresolvability of $X\times\alpha$. For distinct $E,F\in\mathcal{E}$, let $b(E,F)=\{\mu<\alpha\colon E\cap F\cap(X\times\{\mu\})\neq\emptyset\}$. Fix $\kappa<\alpha$ and $\mathcal{D}\in[\mathcal{E}]^\kappa$. Since each $|b(E,F)|<\mathrm{nwd}(X)\leqslant\Delta(X)$ for distinct $E,F\in\mathcal{E}$, $|\bigcup_{E,F\in\mathcal{D}}b(E,F)|\leqslant|\mathcal{D}|\cdot\mathrm{nwd}(X)\leqslant\kappa\cdot\mathrm{nwd}(X)<\alpha$. Fix $\theta\in\alpha-\bigcup_{E,F\in\mathcal{D}}b(E,F)$, then:

(1) $E\cap(X\times\{\theta\})$ is dense in $X\times\{\theta\}$ for every $E\in\mathcal{D}$, and
(2) $E\cap F\cap(X\times\{\theta\})=\emptyset$ whenever $E,F\in\mathcal{D}$ and $E\neq F$.

That is, $\{E\cap(X\times\{\theta\})\colon E\in\mathcal{D}\}$ is a family of pairwise disjoint dense subsets of $X\times\{\theta\}$ of size κ, and since $X\times\{\theta\}$ is homeomorphic to X, we conclude that X is κ-resolvable. □

As we pointed out in the introduction, if X is strongly extraresolvable, then $X \times \alpha$ is strongly extraresolvable for every $\alpha < \mathrm{nwd}(X)$ [7, Theorem 4.1(d')]. It is natural to ask:

Question 2.5. *Is $X \times \omega$ strongly extraresolvable for all strongly extraresolvable space X?*

We are unable to answer this question, but we will give some partial results.

Hrušák [13] has proved that if CH + $2^{\omega_1} < \aleph_{\omega_1}$ holds, then $X \times \omega$ is strongly extraresolvable whenever X is strongly extraresolvable.

Theorem 2.6. *If X contains a family $\{D_\xi\colon \xi < \Delta(X)^+\}$ of dense subsets of X such that*

(1) $|D_\xi \cap D_\zeta| < \omega$ *whenever* $\xi < \zeta < \Delta(X)^+$, *and*

(2) $|\bigcup_{\xi<\theta}(D_\xi \cap D_\theta)| \leqslant \omega$ *for every* $\theta < \Delta(X)^+$,

then $X \times \omega$ is strongly extraresolvable.

Proof. Note that $\Delta(X) = \Delta(X \times \omega)$ and $\omega \leqslant \mathrm{nwd}(X) = \mathrm{nwd}(X \times \omega)$. For each $\theta < \Delta(X)^+$, we enumerate $\bigcup_{\xi<\theta}(D_\xi \cap D_\theta)$ as $\{s(\theta, n)\colon n < \omega\}$, if it is necessary we allow repetition. Now, for each $\theta < \Delta(X)^+$, we define $E_\theta = \bigcup_{n<\omega}[(D_\theta - \{s(\theta, i)\colon i \leqslant n\}) \times \{n\}]$. It is then evident that E_θ is a dense subset of $X \times \omega$ for each $\theta < \Delta(X)^+$. Now if $\mu < \nu < \Delta(X)^+$, then $D_\mu \cap D\nu$ is finite and is contained in $\{s(\nu, n)\colon n < \omega\}$. Hence, $|E_\mu \cap E_\nu| < \omega$ for every $\mu < \nu < \Delta(X)^+$. So, $\{E_\theta\colon \theta < \Delta(X)^+\}$ is a family of dense subsets of $X \times \omega$ such that every two elements of it have finite intersection. Therefore, $X \times \omega$ is strongly extraresolvable. □

Corollary 2.7. *If X is strongly extraresolvable and* $\mathrm{nwd}(X) = \Delta(X) = \omega$, *then $X \times \omega$ is strongly extraresolvable.*

Corollary 2.8. *If X is a countable strongly extraresolvable space, then $X \times \omega$ is strongly extraresolvable.*

Corollary 2.9. *If X is second countable, then $X \times \omega$ is strongly extraresolvable.*

The following result was suggested by the referee.

Corollary 2.10. *If X is strongly extraresolvable and each dense subset is resolvable then $X \times \omega$ is strongly extraresolvable.*

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